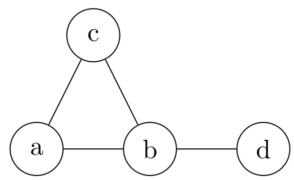
Kaleidoscope Mathematics – Part 2 Exam
Instructor: Daniel Valesin

Duration: one hour. Asking questions during the exam is not allowed.
Your *exam score* is 10 plus the sum of the points you obtain in the questions below, including the bonus question (which is a challenge question that you should leave for last).
Your *exam grade* is equal to the exam score, unless the exam score exceeds 100, in which case your exam grade is 100.

(1) **[15 points]** Consider random walk $X_0, X_1, \dots$ on the graph:



Define the function f on the vertices of the graph by

$$f(\mathrm{a}) = 0.1, \qquad f(\mathrm{b}) = -2, \qquad f(\mathrm{c}) = 8, \qquad f(\mathrm{d}) = 0.$$

Find $\mathbb{E}_{\mathrm{a}}[f(X_2)]$, that is, the expected value of $f(X_2)$ for the walk started from $X_0 = \mathrm{a}$.

(2) **[15 points]** Explain why all states in an irreducible Markov chain have the same period.

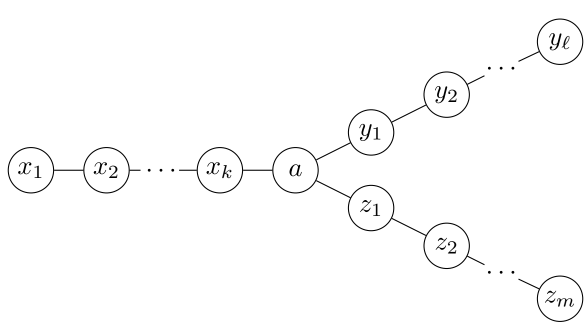
(3) **[20 points]** Prove that, for any two states x and y of a Markov chain, we either have $[x] = [y]$ or $[x] \cap [y] = \varnothing$. You may use the fact that, for any three states a, b, c, if we have $a \rightsquigarrow b$ and $b \rightsquigarrow c$, then $a \rightsquigarrow c$.

(4) **[20 points]** A Markov chain has states space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition matrix P given below (the entry $P(i, j)$ denotes the probability of jumping from i to j, for all $i, j \in S$):

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0.6 & 0 \end{pmatrix}$$

Identify all communicating classes of this chain and give their representation with a directed graph (as done in the lectures: each vertex of the directed graph should represent one communicating class of the chain, and a directed edge from one vertex to another should indicate that it is possible to go from one communicating class to the other with a single jump). Classify all communicating classes as recurrent or transient.

(5) **[20 points]** Consider random walk on the graph represented below (where k, ℓ, m) are positive integers.



Find $\mathbb{P}_a(H_{y_\ell} < H_{z_m})$ (that is, the probability that the walk started from a reaches y_ℓ before reaching z_m). Your answer may depend on k, ℓ, m.

(6) **[Bonus question, 20 points]** For random walk on the graph of Exercise 5, find $\mathbb{P}_a(H_{\{y_\ell, z_m\}} < H_{x_1})$ (that is, the probability that the walk started from a reaches either y_ℓ or z_m before reaching x_1). Your answer may depend on k, ℓ, m.